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The rapid growth and increasing applications of cryptocurrencies are the main factors that drive the cryptocurrency to be considered as a potential asset in investment portfolios. However, recent fierce ups-and-downs, as well as extreme market volatility, have cast doubts on classifying the cryptocurrency as an asset. To investigate the characteristics of cryptocurrencies, we compare Bitcoin, one of the most popular cryptocurrencies, with other major investment assets. Our analysis focuses on the efficient-market hypothesis and the long-term market equilibrium, measured by the Hurst exponent and Shannon entropy, respectively. It is suggested that the bitcoin market is less efficient than other markets, while not significantly different from others in terms of the market equilibrium in the long run. To elucidate these properties, we probe the Fokker–Planck and Schrödinger equations and derive a probability density function, considering the speed of mean reversion and dispersion.

Subject Areas: Complex Systems, Nonlinear Dynamics, Statistical Physics

I. INTRODUCTION

Over the past decade, cryptocurrency markets, including Bitcoin, have experienced tremendous growth. However, trading markets and systems remain unstable and are characterized by extreme volatility and bubbles [1–4]. While cryptocurrencies [5] have a clear influence on the financial industry, their fundamental characteristics, as well as how they are distinguished from other investment assets, have yet to be revealed [2].

Since its introduction, Bitcoin has innovated and challenged monetary and financial systems [6], raising fundamental questions about the meaning of "money" [7]. Therefore, studies comparing Bitcoin with other assets, such as gold, stocks, and the U.S. dollar, are essential for helping policymakers, economists, investors, and other stakeholders understand cryptocurrencies.

The latest on Bitcoin research has mainly provided parsimonious approximations to volatility dynamics and examined scaling exponent, leverage effects, and long-term memory, common among other financial assets [8–11]. Another stream of literature focused on the properties of Bitcoin as a part of financial assets; in particular, studies documented that Bitcoin is similar to gold in terms of scarcity, mining costs, and hedging functions [9, 12, 13]. Some authors opined that Bitcoin is similar to the U.S. dollar since it is both a medium of exchange and a method of transactions [12, 14–16]. Conversely, others reported that Bitcoin is distinct from traditional assets, such as the U.S. dollar and gold [17], due to its highly speculative and unique risk-return characteristics, even including the possibility of price manipulation [18]. In sum, a definitive asset classification of Bitcoin has yet to be established.

Therefore, we set the following research questions to resolve the ambiguity of Bitcoin as an investment asset and examine the nature of its market conditions. How is Bitcoin different from traditional assets in terms of the weak-form efficient market hypothesis (EMH) [19] and in relation to the overall market (system) equilibrium? In addition, what are the theoretical fundamentals explaining the differences between Bitcoin and traditional assets? The (weak-form) EMH is related to the *characteristics* of the market whether asset prices reflect all available (price) information, while the equilibrium indicates the *state* of a market in which all competing influences are balanced. Thereby, these questions are well expected to deepen our fundamental understandings of the cryptocurrency markets represented by Bitcoin.

In this study, we conduct a time-series analysis of traditional assets, specifically, gold, the U.S. dollar, and a stock index (S&P 500) and compare the results with those of Bitcoin. We first examine market efficiency and collective phenomena through the use of Hurst exponent (HE) and a scaling exponent, and carry out a symbolic time-series analysis (STSA) to probe the dispersion of dynamic rise-fall patterns by means of the Shannon entropy. We further derive a probability density function (PDF) evolving in time, which explains the relationship among the market efficiency, collective phenomena, and long-term equilibrium in association with the speed of mean reversion and dispersion.

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Ultimately, this paper establishes a theoretical basis for revealing market characteristics not yet clarified in literature. Existing studies mainly focused on return volatility, an approach that emphasizes ex-post analysis. Unlike such studies, we conclude that the dispersion of probability allocation in each state, namely entropy [20], is different from general expectations, such as return volatility and market efficiency. We further explain theoretically the market efficiency, long-term equilibrium, and scale invariance property of Bitcoin and other selected assets, deriving a time-dependent PDF that reflects the speed of mean reversion and dispersion in the markets.

This paper is organized as follows: Sec. II lays out the theoretical framework. In Sec. III, the methodology and the data are presented, whereas Sec. IV discusses the result. Finally, a conclusion is given in Sec. V.

II. THEORETICAL FRAMEWORK

In this section, we follow Ahn et al. [21] to derive Laplace distribution of asset returns from the Schrödinger equation, which is obtained from the Fokker–Planck (FP) equation. The model also predicts the connection between the power-law exponent (PLE) and the speed of mean reversion, a proxy for market efficiency.

Asset returns are defined as follows:

$$x \equiv \ln\left(\frac{p_{t+\delta t}}{p_t}\right),$$

where p_t represents the asset price at time t. The time evolution of the asset returns is governed by the stochastic differential equation (SDE):

$$dx = v(x,t)dt + \sigma(y,t)dW_t,$$

where W_t is a standard Wiener process. We assume that the drift of asset returns arises from an external potential $V(x,t)$ in the following manner:

$$v(x,t) = -\frac{\partial V(x,t)}{\partial x} \equiv -V_x,$$

which is analogous to classical mechanics. We further define the diffusion coefficient

$$D(x,t) \equiv \frac{1}{2}\sigma^2(x,t)$$

and express the above SDE as the FP equation for the PDF $\rho(x,t)$:

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{\partial^2}{\partial x^2}[D(x,t)\rho(x,t)] + \frac{\partial}{\partial x}[V_x\rho(x,t)]. \quad (1)$$

For simplicity, we also assume that the diffusion coefficient is a constant, i.e., $D(x,t) = D$. Then Equation (1) takes the concise form:

$$\frac{\partial}{\partial t}\rho(x,t) = \hat{L}\rho(x,t), \quad (2)$$

where the FP operator reads $\hat{L} = V_{xx} + V_x\frac{\partial}{\partial x} + D\frac{\partial^2}{\partial x^2}$.

To solve Equation (2), we first examine its steady-state solution $\rho_s(x)$, satisfying $\hat{L}\rho_s(x) = 0$. A simple functional form of $\rho_s(x)$ is given by Putz [22]:

$$\rho_s(x) = \frac{1}{C}\exp\left(-\frac{V(x)}{D}\right),$$

where $C \equiv \int_{-\infty}^{+\infty}\exp\left(-\frac{V(x)}{D}\right)dx$ is the normalization constant. We now introduce the "wave function" $\Psi(x,t)$ and Hermitian operator $\hat{H}$ via

$$\Psi(x,t) \equiv \frac{\rho(x,t)}{\sqrt{\rho_s(x)}}$$

$$\hat{L}\rho(x,t) \equiv -\sqrt{\rho_s(x)}\hat{H}\Psi(x,t),$$

which yields $\hat{H} = -\frac{1}{2}V_{xx} + \frac{1}{4D}V_x^2 - D\frac{\partial^2}{\partial x^2}$. Then, with the imaginary time $\tau \equiv -i\hbar t$ and mass $m \equiv \frac{\hbar^2}{2D}$, Equation (2) can be rearranged into the Schrödinger equation

$$\begin{aligned} i\hbar\frac{\partial}{\partial\tau}\Psi(x,\tau) &= \hat{H}\Psi(x,\tau) \\ &\equiv \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)\right)\Psi(x,\tau), \end{aligned} \quad (3)$$

where the effective potential $U(x)$ is given by

$$U(x) = -\frac{V_{xx}}{2} + \frac{V_x^2}{4D}.$$

Financial time series tend to exhibit a mean reversion [23]. There is literature documenting contrarian effects in stock markets around the world [24–26]. Relatively high or low asset returns will revert to the equilibrium, indicating that an external potential stabilizes short-term fluctuations toward the long-term equilibrium. We thus take $V(x) = \alpha|x-a|$ as a simple model for the potential: If asset returns deviate from the equilibrium return a, the market field will draw the asset returns back to a at the constant speed of α.

One plausible explanation about the mean-reverting features of asset returns is the divergence between fundamental and market values, which causes arbitrage trading. Speculative investors eventually eliminate the differences between the two [27]. Owing to the leptokurtic features of asset returns and mean reversion driven by market forces, asset returns mostly remain in sidewalk markets [28]. While Ahn et al. [21] proposed a harmonic oscillator to model this potential, we define it here to have a constant speed of mean reversion. The return distribution modeled with a quantum harmonic oscillator has a tail that diminishes in proportion to $\exp(-x^2)$, which is too fast when Bitcoin returns are severely leptokurtic and highly volatile. It is thus not quite conceivable that a larger market force affects the returns toward the long-term mean level when they are moving away from it. Therefore, we assume that the restoring speed from realized returns to the long-term equilibrium is constant.

Given $V(x) = \alpha|x-a|$, we have the effective potential

$$U(x) = -\alpha\delta(x-a) + \frac{\alpha^2}{4D},$$

where $\delta(\cdot)$ is a Dirac delta function, and the extra drift $\frac{\alpha^2}{4D}$ does not affect the wave function [29]. We thus obtain the following time-independent Schrödinger equation

$$\left(E - \frac{\alpha^2}{4D}\right)\psi(x) = \hat{H}\psi(x) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) - \alpha\delta(x-a)\psi(x),$$

which can be solved easily. Specifically, the ground-state solution is given by

$$\psi_0(x) = \frac{\sqrt{m\alpha}}{\hbar}\exp\left(-\frac{m\alpha}{\hbar^2}|x-a|\right),$$

with energy $E = E_0 \equiv -\frac{m\alpha^2}{2\hbar^2} + \frac{\alpha^2}{4D} = 0$ [30], whereas other solutions describing continuum states read, apart from normalization,

$$\psi_k(x) = e^{ik(x-a)} + \frac{m\alpha}{m\alpha - i\hbar k^2}e^{ik|x-a|}$$

with energy $E = E_k \equiv \frac{\hbar^2 k^2}{2m} + \frac{\alpha^2}{4D} > 0$. Accordingly, Equation (3) bears the general solution of the form

$$\Psi(x,\tau) = A\psi_0(x)e^{-\frac{iE_0\tau}{\hbar}} + \textstyle\int dk A(k)\psi_k(x)e^{-\frac{iE_k\tau}{\hbar}}.$$

The PDF then takes the form

$$\begin{aligned}\rho(x,t) &= \sqrt{\rho_s(x)}\Psi(x,\tau=-i\hbar t)\\ &= \sqrt{\rho_s(x)}[A\psi_0(x) + \textstyle\int dk A(k)\psi_k(x)e^{-E_k t}],\end{aligned} \quad (4)$$

which reduces in the asymptotic limit ($t \to \infty$) to

$$\begin{aligned}\rho(x) \equiv \rho(x, t\to\infty) &= \sqrt{\rho_s(x)}A\psi_0(x)\\ &= \sqrt{\frac{\alpha}{2D}}A\exp\left(-\frac{\alpha}{D}|x-a|\right).\end{aligned}$$

Putting the normalization constant A, we finally obtain the PDF corresponding to a Laplace distribution:

$$\rho(x) = \frac{\alpha}{2D}\exp\left(-\frac{\alpha}{D}|x-a|\right). \quad (5)$$

From Equation (5), we can also obtain the tail distribution of asset returns. Defining the gross return $Y \equiv \frac{p_{t+\Delta t}}{p_t} = e^x$, we have, for the right tail satisfying $y > e^a$,

$$P(Y \geq y) = P(x \geq \ln y) = \int_{\ln y}^{+\infty} \frac{\alpha}{2D}e^{-\frac{\alpha}{D}|x-a|}dx \approx y^{-\frac{\alpha}{D}} \quad (6)$$

which follows the power-law distribution with the exponent $\frac{\alpha}{D}$. Furthermore, the entropy of a Laplace distribution is given by [31–33]:

$$H(x) = \ln\left(\frac{2D}{\alpha}\right) + 1. \quad (7)$$

Equations (4) to (7) lead us to summarize the dynamic characteristics of asset returns as follows: (i) In short times, Equation (4) manifests that time t affects the exponent, and the PDF varies following α^2. As shown in Fig. 1, the higher the speed α of the mean reversion process, the greater the sensitivity to changes in t. (ii) In the case of long-time dynamics, described by Equation(5), the exponential part of the Laplace distribution changes with the relative values of α and D. In particular, when α is larger than D, the entire distribution becomes steeper. (iii) The right tail of the Laplace distribution in Equation (6) is characterized by the PLE, given by the ratio of α to D. Accordingly, the estimated values of the PLE can be used for demonstrating the EMH. For instance, if the estimated value of the PLE of a particular asset increases, either α should increase or D should decrease. A decrease of α, in particular, can be interpreted as an enhancement in the overall market efficiency. (iv) Finally, since the entropy of a Laplace distribution can be expressed as a linear function of volatility [31, 33], we can estimate the market equilibrium with the relationship between the Laplace exponent and entropy, given by Equation (7).

III. EMPIRICAL ANALYSIS

A. Hurst Exponent

We first analyze the rescaled adjusted range statistics, denoted by $(R/S)_n$, following Hurst [34, 35] and Mandelbrot and Wallis [36, 37]:

$$(R/S)_n = cn^H,$$

where n is the length of the fractioned time series, c is a constant, and H is the HE. Taking the logarithm, we can estimate the HE as the slope of the data fitted to

$$\log(R/S)_n = \log c + H\log n.$$

The R/S statistics and the standard deviation S_n are determined as follows:

$$(R/S)_n = \frac{1}{S_n}\left[\max_{1\leq t\leq n}\sum_{i=1}^{t}(r_i - \bar{r}_n) - \min_{1\leq t\leq n}\sum_{i=1}^{t}(r_i - \bar{r}_n)\right]$$

$$S_n = \left[\frac{1}{n}\sum_{i=1}^{t}(r_i - \bar{r}_n)^2\right]^{1/2},$$

where t is a specific point in the time series, r_i is the return at time i, and $\bar{r}_n$ is the average of time-series returns divided by length n.

Ranging from 0 to 1, the HE H measures the market efficiency, long-term memory, and fractality of time series [38]. Based on the estimated value of H in the above equation, the diffusion process of time series can be classified into the following three categories: (i) a random walk for $H = 0.5$, interpreted here as a special case

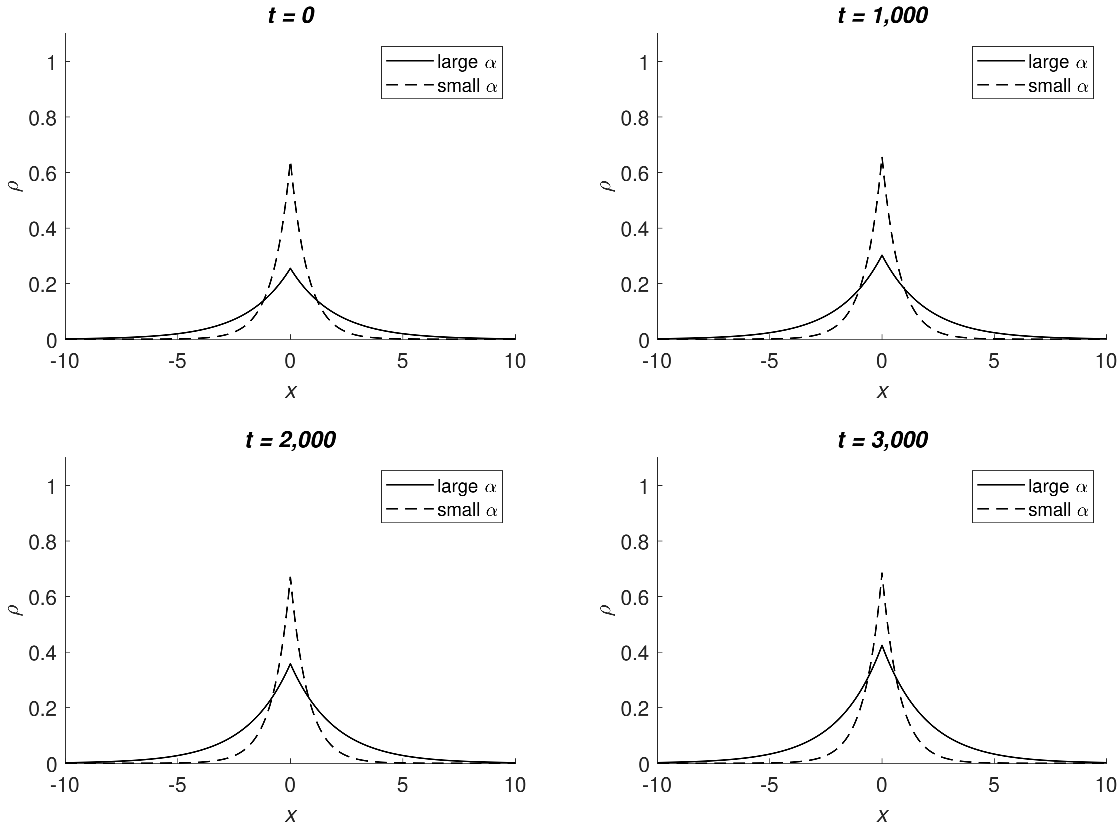

FIG. 1. Plots of the PDF for Bitcoin (solid lines) and gold (dashed lines) at time t = 0, 1000, 2000, and 3000, based on Equation (4) for given α and D. Bitcoin has a large value of α ($\sim 10^{-3}$), and the other assets including gold have small values of α ($\sim 10^{-5}$). Since those other assets are similar in the PDF, only the PDF of gold is presented among them.

of fractional Brownian motion; (ii) an anti-persistent series for $0 < H < 0.5$, representing the mean-reverting property; and (iii) a persistent series for $0.5 < H < 1$, exhibiting long-term positive autocorrelations.

B. Power Law

We estimate the PLE of log returns to evaluate the collective phenomenon in terms of the scaling exponent in the markets. The PLE is estimated from the following counter-cumulative density function (also known as a survival function):

$$1 - F(x) = P(X > x) \propto x^{-\xi},$$

where $F(x)$ is a cumulative density function and ξ is the PLE. Taking the logarithm on both sides of the above equation, we obtain the following linear relationship:

$$\ln P(X > x) = c - \xi \ln x + \epsilon,$$

where c is a constant and ϵ is an error term following the normal distribution (*i.i.d.*). The PLE is usually estimated as the slope of the above equation. The estimated PLE has an asymptotic standard error of $\hat{\xi}(m/2)^{-1/2}$, where m is the number of observed samples [39].

The commonly used ordinary least-squares (OLS) method may yield inaccurate PLE estimates. In some cases, where the OLS estimation is accurate, the results can be unsatisfactory, as there is no indication of whether the data obey a power law. Additionally, OLS estimates are subject to the following systematic and potential errors: (i) OLS cannot describe errors accurately in the log histogram for power-law analysis; (ii) OLS can produce high R^2 values, even if the actual form of the distribution does not follow a power law; and (iii) OLS-based power-law estimation does not satisfy the basic requirements of a probability distribution, such as normalization [40]. We therefore estimate the PLE based on the Kolmogorov–Smirnov (KS) statistics as well.

C. Entropy

Finally, we calculate the STSA entropy of discrete random variables. According to Shannon [41], the entropy

TABLE I. Descriptive statistics of daily log returns. Observations are made on a total of 2,053 trading days among 3,014 days in the sample period from October 1, 2010 to December 31, 2018.

	Mean	Min.	Max.	Std.	Skewness	Kurtosis
Bitcoin	5.29×10^{-3}	-1.04	1.00	0.07	0.43	50.53
USD/EUR	-5.45×10^{-5}	-0.03	0.03	0.01	-0.01	4.71
Gold	1.24×10^{-5}	-0.10	0.05	0.01	-0.57	10.04
S&P 500	4.02×10^{-4}	-0.07	0.05	0.01	-0.59	7.87

of the discrete random variable X is given by

$$H(X) = -\sum_{i=1}^{M} p(x_i) \ln p(x_i),$$

where M is the number of possible outcomes of the random variable X and $p(x_i)$ is the probability that each outcome of experiments is assigned to $X = x_i$. Since entropy measures the dispersion of the probability allocation rather than that of observed outcomes, entropy is more robust to extreme observations [31]. Typically, entropy is the highest when the randomness of the system is maximal and lowest when randomness is minimal or complete information is given. In this fashion, entropy provides better information about the underlying distribution of random variables and serves as a more appropriate measure of uncertainty than volatility [33, 42, 43].

Moreover, since the STSA method is resistant to noise, it is widely applied in physics, information theory, and finance [33, 44]. Following Ahn et al. [33], we symbolize the real values of data into a series of sequence bundles composed of binary numbers. Each sequence consists of the consecutive returns of an asset S, expressed as 1 for positive returns and 0 otherwise. It is then converted to a decimal number X^S. Applying such transformation to all the sequences on a daily basis, we finally write the Shannon entropy of the discrete variable X^S in the following form:

$$H(X^S) = -\sum_{i=1}^{M-S} p(x_i^S) \log_2 p(x_i^S).$$

To make up for the increase of $H(X^S)$ with S, we further normalize the Shannon entropy as follows:

$$h(X^S) = \frac{1}{S} H(X^S).$$

In this paper, we take $S = 3$ (robustness for $S = 2$ and 4 has been checked as well) and, by means of the STSA, identify the time-varying patterns in log returns. Ruiz et al. [45] and Ahn et al. [33] reported that the STSA entropy better captures uncertainty in a financial time series than the histogram-based entropy.

D. Data

To compare the characteristics of each asset, we refer to daily data for Bitcoin (price), gold (price), the U.S. dollar (USD/EUR exchange rate), and the stock index (S&P 500). Bitcoin prices have been collected from Quandl.com, and gold prices retrieved from the World Gold Council. USD/EUR exchange rates and S&P 500 data have been obtained from the Federal Reserve Bank of St. Louis. Bitcoin exchanges never close, and we have sampled the Bitcoin price from October 1, 2010 to December 31, 2018 (3,014 days). On the other hand, data for the other three assets are available only on trading days. Accordingly, we have matched Bitcoin data with the data for the other three assets. As a result, we have obtained 2,053 observations of daily data for each of the four assets.

As shown in Table I, the mean, maximum, minimum, and standard deviation of Bitcoin's daily log returns are relatively large compared with those of the other asset classes [2, 46–48]. The return distribution of other assets is characterized by negative skewness, which implies investors' risk-averse attitudes [49, 50]. Bitcoin, however, shows positive skewness, similar to emerging stock markets. Bitcoin also has a more leptokurtic feature than others, which leads to a heavy tail in its return distribution and provides some clues about collective behavior [51]. Moreover, all the time-series data are nonstationary; we thus take the difference in log prices, which are stationary, for further analysis.

As shown in Fig. 2, the daily log returns of Bitcoin, as well as other assets, are described well by a Laplace distribution rather than a Gaussian distribution [55–59]. The Laplace distribution is leptokurtic, allocating the probability distribution more at the peak and tails, and displays decay at both tails slower than a normal distribution resulting in scaling phenomena. Its entropy reaches the maximum under specific constraints on the dispersion [32]. The fact that all the four assets well follow Laplace distributions thus implies that they share certain market characteristics related to the scaling exponent and entropy.

IV. RESULTS AND DISCUSSION

Considering both HE and PLE, we conclude that the bitcoin market is less efficient than other asset markets. With regard to the EMH [60, 61], Bitcoin ($H > 0.5$) is distinguished from the other assets ($H \simeq 0.5$), which resemble a random walk, and our findings are generally in line with Bariviera et al. [62], Bariviera [63], and Al-

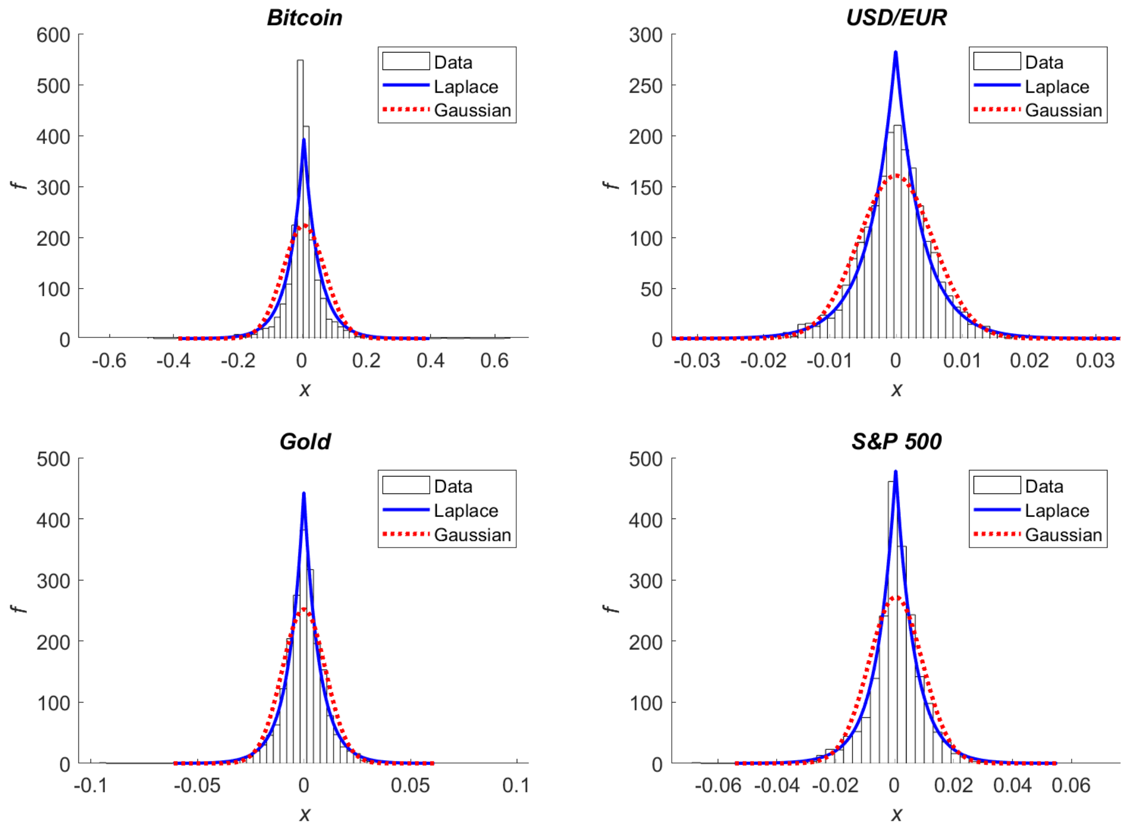

FIG. 2. Histogram of log returns. The optimal bin size of the histogram has been taken to be the mean of the values calculated in the way following Sturges [52], Scott [53], and Freedman et al. [54]. Each asset appears to follow a Laplace distribution rather than a Gaussian distribution.

Yahyaee et al. [64]. In particular, as shown in Table II, the bitcoin market demonstrates long-term memory effects ($H \approx 0.60 \pm 0.04$) and the market efficiency is lower due to its self-referential nature [38, 65]. It is further confirmed by the PLE of Bitcoin (taking the value approximately 0.50), appreciably smaller than that of the other assets ($\xi \geq 1$), in the right tail of the distribution. We consider such a low value of the PLE to be a signature for important phenomena with regard to market efficiency [66]. Note that the fractal structure in time series indicates the underlying presence of herding behavior: In particular, the change in the PLE implies a transition from efficient market trading to herding behavior [67]. Accordingly, the relative inefficiency of the bitcoin market is observed in both the entire sample (HE) and the right tail (PLE).

The Laplace distribution is characterized by allocating more probability at the peak and tails compared with its counterpart, i.e., Gaussian distribution. Considering the mean reversion speed α and the diffusion coefficient D, which collaborate on forming the peak and tails of a certain distribution together, we could conjecture the degree of market efficiency. As can be seen in Fig. 3, the conditional volatility of Bitcoin is considerably high compared with other assets and results in a high diffusion coefficient [31]. The market efficiency closely relates to the rate of diffusion of a series [68] in comparison with the rate of diffusion in a geometric Brownian motion (GBM). Therefore, the diffusive feature of the bitcoin market primarily contributes to more probability allocation at the tails of the distribution, resulting in low market efficiency. Moreover, the mean reversion speed of Bitcoin, much faster than the other assets, further supports our findings. The higher value of α implies more probability allocation at the peak, compared with its benchmark, i.e., GBM whose α is defined to be 0 in Equation (6). Put differently, the slow speed of the mean reversion (i.e., the smaller value of α) in log returns implies the price process is close to a random walk in our setting. Thereby, the bitcoin market is concluded inefficient compared with the others as it exhibits more leptokurtic features with large α and D.

On the other hand, Bitcoin and the other assets are similar in the market's long-term equilibrium of dynamic rise and fall patterns. As can be seen in both Table III and Fig. 4, there is no significant difference between the STSA entropy of Bitcoin and that of other assets.

TABLE II. Summary of market efficiency and scale invariance. To confirm the robustness, we estimate both the classical and the corrected HE. The mean reversion speed α has been obtained via multiplying the PLE by the dispersion of each asset ($\alpha = \xi D$). Additionally, we find that both the OLS and KS methods do not yield significantly different values of the PLE.

	Hurst Exponent		Speed of Mean Reversion	Power-Law Exponent	
	Classical	Corrected		OLS	KS stat
Bitcoin	0.67 ± 0.04	0.60 ± 0.04	$(1.33 \pm 0.05)\times 10^{-3}$	0.51 ± 0.06	0.50 ± 0.02
USD/EUR	0.57 ± 0.04	0.51 ± 0.03	$(2.27 \pm 0.04)\times 10^{-5}$	1.45 ± 0.17	1.36 ± 0.02
Gold	0.55 ± 0.04	0.48 ± 0.02	$(6.66 \pm 0.08)\times 10^{-5}$	1.28 ± 0.15	1.14 ± 0.02
S&P 500	0.52 ± 0.04	0.45 ± 0.03	$(5.38 \pm 0.09)\times 10^{-5}$	1.30 ± 0.15	1.22 ± 0.02

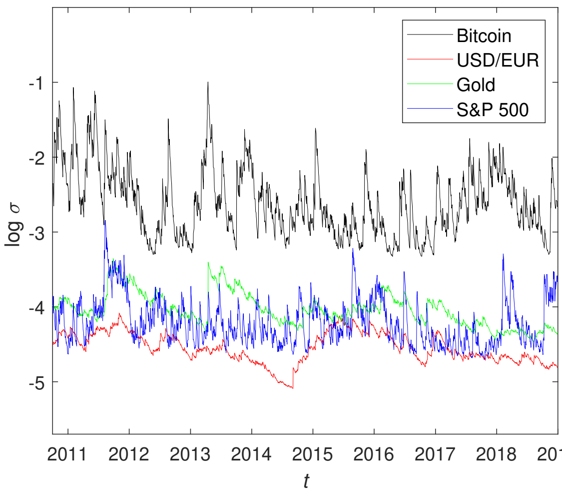

FIG. 3. The conditional volatility, computed via the GARCH(1,1) model from daily log returns for each asset class, is presented on the log scale. Bitcoin is around 40 times more volatile than the other assets, which remain relatively constant over time.

This is somewhat odd: The entropy of the bitcoin market might be significantly different from other markets because the Shannon entropy is known to have a linear relation with volatility for a particularly well-defined distribution [31, 33]. However, the STSA entropy, unlike the entropy constructed out of raw data (quantile or histogram-based), reflects the dynamic rise and fall patterns of several consecutive returns rather than the variation of a single return [33, 69], which implies not a simple linear transformation of volatility. As explained by Equation (7), the even quantile or histogram-based entropy becomes large or small by the composition of α and D together. In particular, the internal dynamics of the market can fluctuate according to the mean reversion speed and volatility, implying that long-term market equilibrium might be different depending on the composition of α and D, apart from the market efficiency.

As shown in Fig. 5, the scaling exponent of the return distribution in the bitcoin market tends to increase from 0.62 (2011) to 1.79 (2018) and results in the collective behavior of degree similar to other markets [70]. Each es-

TABLE III. Monthly STSA entropy for each asset from October 1, 2010 to December 31, 2018. The entropy of Bitcoin does not appear to be significantly different from that of other assets.

	Mean	Mode	Median	Min.	Max.
Bitcoin	0.87	0.83	0.87	0.55	0.98
USD/EUR	0.86	0.89	0.88	0.52	0.98
Gold	0.86	0.89	0.88	0.57	0.98
S&P 500	0.86	0.97	0.87	0.61	0.98

timated value of the PLE in the bitcoin market is located in the range of the Levy-stable region (colored gray) and below the decay limit (designated by the horizontal dotted line) like other assets. At the same time, both α and D have decreased and resulted in the relocation at the peak and tails, closer to Gaussian. Therefore, α and D, individually and together, have improved the overall efficiency of the bitcoin market, synchronizing with other markets [71, 72]. Our findings also explain well the controversy in previous studies, as to whether the bitcoin market becomes efficient or not: acceptance or rejection of EMH over time [62, 73–78].

In Sec. II, from the FP and Schrödinger equations, we have shown that the PDF of log returns under the potential of $V(x) = \alpha|x - a|$ results in a Laplace distribution. The market efficiency could be well characterized by the expression $\frac{\alpha}{D}$ (> 0), which corresponds to the scale parameter of the PDF: (i) Bitcoin has a relatively larger value of dispersion; (ii) the mean reversion speed of the Bitcoin return is much faster than that of the other assets; and (iii) Bitcoin has a smaller value of the PLE than other assets as the faster mean reversion speed cannot catch up the larger value of dispersion. In other words, the overall effects of the scaling exponent and mean revision speed together explain well the market efficiency of the bitcoin market. We postulate that non-commercial speculative trading [18] might lead to low complexity in the bitcoin market and create self-sustaining dynamics having positive feedback, resulting in market inefficiency due to slower decay at the tail and faster mean-reverting speed at the peak. However, if the entropy of the system is not maximized and the components of the system interact with each other directly or indirectly, then any

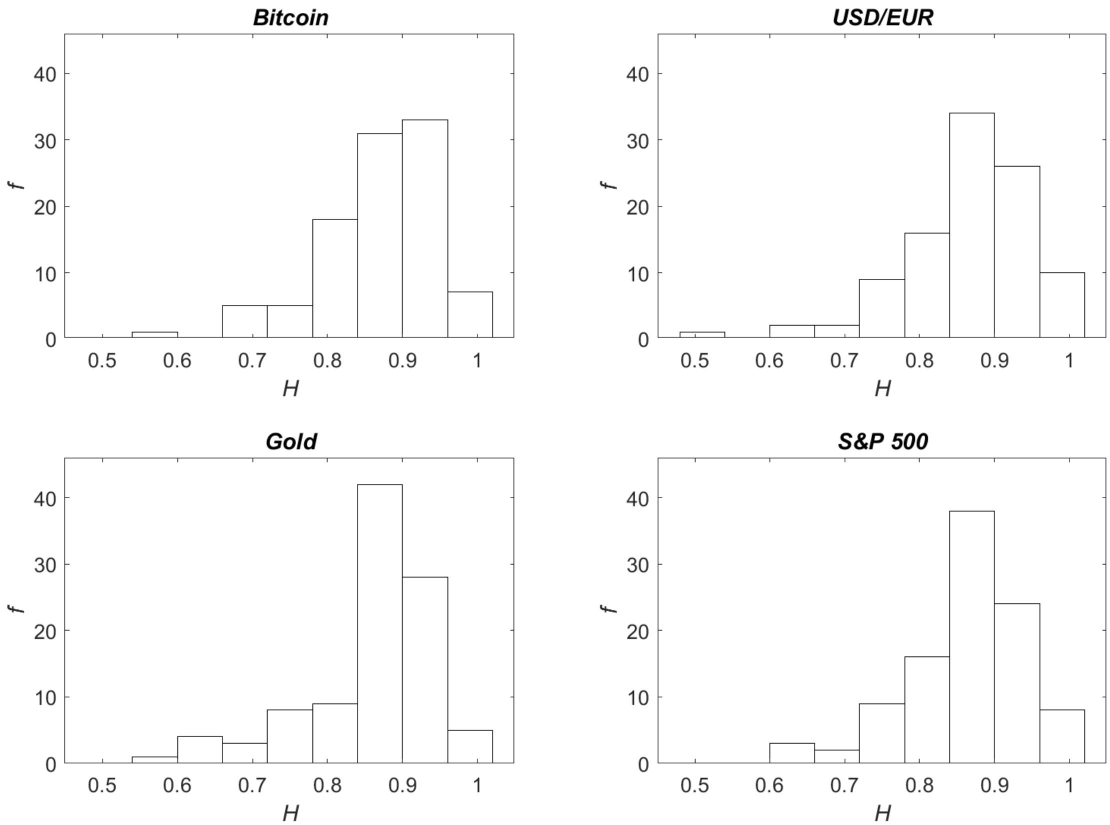

FIG. 4. The STSA entropy for each asset has been obtained from daily rise-fall patterns with the bin size of 0.06. We have also varied the bin size from 0.05 to 0.07 to confirm the robustness of our results.

path that allows the entropy to increase will be finally realized. Then the log return distribution derived from the mean-reverting potential $V(x)$ should ultimately proceed toward a distribution that maximizes the entropy, i.e., a Laplace distribution. Therefore, in the current state, the degree of the market equilibrium is not significantly different among the assets.

V. CONCLUSION

We find that the bitcoin market is currently at an intermediate stage of development, not consistent with the EMH. However, with respect to the spread of the probability assigned to each state, the bitcoin market is not far from equilibrium in comparison with other assets. Ultimately, our findings underscore the relative immaturity of the bitcoin market, described well by a Laplace distribution, with the constant restoring speed from realized returns to the long-term equilibrium.

This study not only presents the unique position of Bitcoin as an asset but also provides a new understanding of market efficiency. In particular, the probability distribution and its relocation at the peak and tails would determine the degree of market efficiency. On the other hand, in terms of market maturity, while the bitcoin market has not yet matured, its properties are not too far from that of others. In addition, the relationship between complexity and entropy might provide a better understanding of the future of cryptocurrencies, which is left for further study.

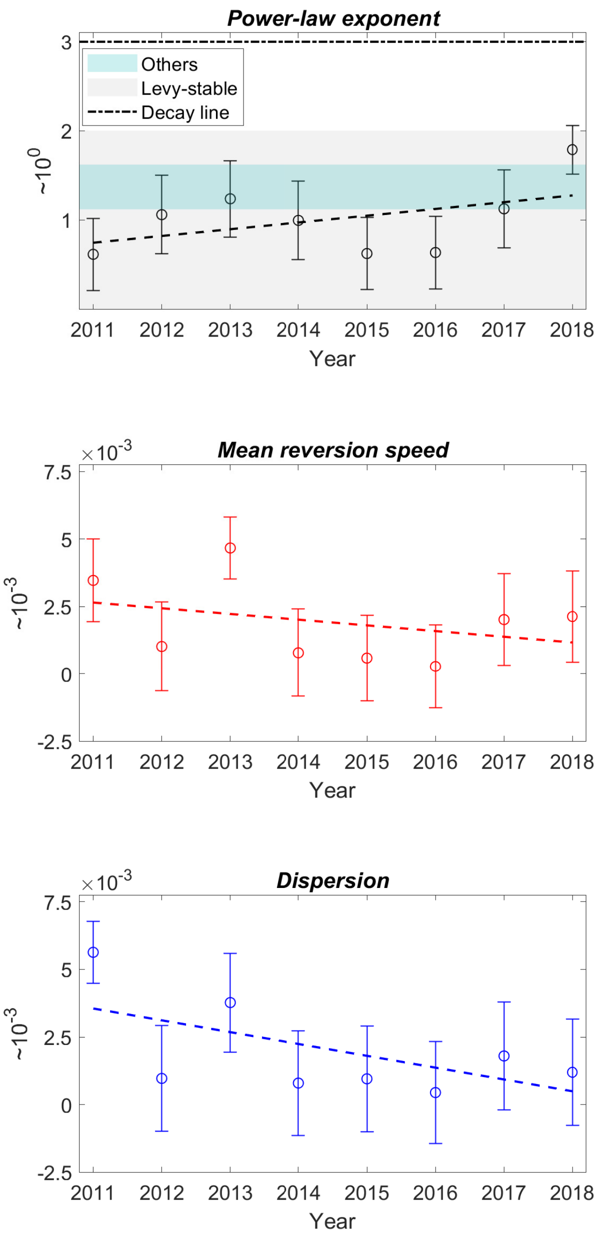

FIG. 5. Evolution of the bitcoin market properties. The estimated values of the PLE are marked with an empty circle in the regression line (black), where each error bar corresponds to one standard deviation. They are located within the Levy-stable region colored gray. The blue zone represents the range of the PLE of the other assets, including one standard deviation. Calculated values of the mean reversion speed α and dispersion D are also plotted with an empty circle in the regression lines (red and blue, respectively), where each error bar corresponds to one standard deviation.

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[5] "Virtual currency" is defined as a "type of unregulated money, which is issued and usually controlled by its developers and used and accepted among the members of a specific virtual community" [80]. "Digital currency" is a form of virtual currency that is electronically created, stored, and expressed. Digital currency, as a separately defined currency unit, is different from the digital payment mechanism of electronic money (e-money), including fiat money. "Cryptocurrency," a subset of digital currency, relies on cryptographic techniques to achieve agreement about a currency's value. "Bitcoin," a cryptocurrency, uses a peer-to-peer network. Although Bitcoin was not the first digital currency, it is regarded as the most successful, since its inception to achieving a 32% to 95% share of all digital currencies' total market capitalization [85, 87, 88].

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[19] With the EMH, since asset prices quickly reflect all available information, market prices are impossible to predict. An efficient market, in which all available information is fully reflected in asset prices, is divided into three categories: "weak," "semi-strong," and "strong," according to the range of information provided [60, 61]. The EMH used in this study is the weak-form one, which addresses that market prices may not be predicted because all currently available price information is already reflected in asset prices. Following Fama [61], many authors have studied the efficiency of the bitcoin market from the perspective of the EMH, one of the fundamental assumptions in finance. They showed discordant results that the bitcoin market is efficient or not: Some studies argued that the bitcoin market has no or less efficient character [64, 74, 77] while others mentioned it becomes efficient over time [8, 63, 73, 75]. In addition, market efficiency has been changed by several internal and external conditions that affect the bitcoin market [76, 78]. In brief, there are still debates about the efficiency in the bitcoin market.

[20] Using entropy measures, we demonstrate that Bitcoin is not significantly different from traditional assets in terms of the market's long-term balance or its degree of equilibrium. In particular, entropy represents how close the system is to equilibrium: the higher the entropy, the greater the disorder. Unlike the conventional notions of equilibrium in economics, the term 'statistical equilibrium' has been widely used in physics and information theory representing the most likely state of the system in the form of a probability distribution that can be derived by maximizing the entropy of the system. Thus, the entropy aims to find a distribution of a non-degenerated probability of the target system that accounts for the central tendency and the fluctuations around [81, 82, 89].

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